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Solutions of Maxwell's Equations for Charge Moving with the Speed of Light

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Abstract

It is shown that Maxwell's equations admit solutions for charge moving with the speed of light. These are globally regular, but if the charge is one of sign only, the total energy is infinite. However, if equal amounts of positive and negative charge are present, the total energy can be finite, and such solutions seem physically unobjectionable.

1. *Introduction*

Plane wave solutions of Maxwell's equations are source-free, and physically this is taken to mean that the sources occur at infinity. A generalisation of plane waves is that class of waves whose fronts are plane but whose amplitudes vary over each front: these are sometimes called *plane-fronted waves* (Kundt, 1961). Among the plane-fronted waves there is a large sub-class which have sources in the finite part of space–time, and these are investigated here. The sources are shown in Section 2 to be charges travelling with the speed of light c.

One can construct a model for a charged particle moving with speed c (Section 3), but this is unphysical because its total energy is infinite. If, however, we take a dipole particle the total energy may be finite, as is shown in Section 4.

Whether the fields arise from purely retarded contributions of the source is by no means obvious, and this is investigated in the Appendix.

The only relevant previous work I have been able to find is that of Bateman (1915), who studied fields of singularities moving with speed c from a rather general point of view, but without considering the nature of the source or the field energy; and some brief remarks of Sommerfeld (1905).

2. *The Electromagnetic Field*

I use throughout the metric of special relativity

$$ds^2 = -dx^2 - dy^2 - dz^2 + dt^2 \tag{2.1}$$

choosing unrationalised Gaussian units but with the unit of time chosen so that the velocity of light, c, is 1. The coordinates are numbered

$$x^1 \equiv x, \qquad x^2 \equiv y, \qquad x^3 \equiv z, \qquad x^4 \equiv t$$

The field will be generated by a vector potential

$$A^i = (0, 0, \phi, \phi) \qquad \text{or} \qquad A_i = (0, 0, -\phi, \phi) \tag{2.2}$$

where ϕ is a function of differentiability class C^1 and piecewise C^2 on every closed interval of the x^i. This ensures the continuity of the field F_{ik} and precludes surface charges and surface currents. The function ϕ will be given the form

$$\phi = \phi(x, y, u), \qquad u \stackrel{\text{def}}{=} t - z \tag{2.3}$$

and F_{ik}, given by

$$F_{ik} = A_{i,k} - A_{k,i} \tag{2.4}$$

(comma denotes partial differentiation) becomes

$$F_{ik} = \begin{pmatrix} 0 & 0 & \phi_x & -\phi_x \\ 0 & 0 & \phi_y & -\phi_y \\ -\phi_x & -\phi_y & 0 & 0 \\ \phi_x & \phi_y & 0 & 0 \end{pmatrix} \tag{2.5}$$

where a subscript x or y means $\partial/\partial x$ or $\partial/\partial y$. The four current

$$J^i = (4\pi)^{-1} F^{ik}{}_{,k} \tag{2.6}$$

has components

$$J^i = (0, 0, \rho, \rho) \tag{2.7}$$

where

$$4\pi\rho = -\nabla^2 \phi \equiv -(\phi_{xx} + \phi_{yy}) \tag{2.8}$$

so that in empty space, where $J^i = 0$, ϕ satisfies

$$\phi_{xx} + \phi_{yy} = 0 \tag{2.9}$$

The electromagnetic field (2.5) is *null*, i.e. it satisfies

$$F_{ik} F^{ik} = 0 \qquad \text{and} \qquad \eta^{ijkm} F_{ij} F_{km} = 0 \tag{2.10}$$

It is often called a *plane-fronted wave*, because the wave-fronts are simply $z - t = \text{const.}$ A plane wave is a special case of a plane-fronted wave in which the F_{ik} are constant over a wave-front. The energy tensor, which because of (2.10) reduces to

$$E_k{}^i = -F^{ia} F_{ka} \tag{2.11}$$

has the following non-zero components

$$-E_3{}^3 = -E_3{}^4 = E_4{}^3 = E_4{}^4 = \phi_x{}^2 + \phi_y{}^2 \stackrel{\text{def}}{=} W \tag{2.12}$$

Fix attention on a particular Lorentz frame S, and suppose that at the world-point P the source consists of identical charges e. Then

$J^4 = e \times$ number of charges per unit volume in S;
$J^3 = e \times$ number of charges crossing unit area in unit time in S,

and the equality $J^3 = J^4$ in (2.7) implies that the charges are moving with unit velocity. Hence *the sources consist of charge of density* ρ *moving with the speed of light*. This is not surprising, because J^i is a null vector, i.e. $J^i J_i = 0$. However, the assumption that at every point P charge of only one sign is present is essential.† Nevertheless, ρ can assume different signs at different world-points.

Except for solutions depending solely on u, every solution of Laplace's equation (2.9) must be singular. I shall now show how a singularity in the finite region of the x, y plane may be replaced by a region in which a current is flowing, and in which F_{ik} are continuous. Suppose ϕ is a function [e.g. $\phi = x(x^2+y^2)^{-1}\exp(-\frac{1}{2}u^2)$] whose only singularities lie on the 2-surface in Minkowski space–time

$$S: \qquad x = 0, \qquad y = 0 \tag{2.13}$$

To obtain a globally regular field we surround S by the 3-surface

$$\Sigma: \qquad x^2 + y^2 = a^2 \tag{2.14}$$

a being a positive constant, and replace ϕ by some function ϕ^* of class C^2 inside and on Σ, that is, in

$$D: \qquad \{(x, y, z, t): x^2 + y^2 \leqslant a^2\} \tag{2.15}$$

we also require that

$$\phi^* = \phi, \qquad \frac{\partial \phi^*}{\partial x^i} = \frac{\partial \phi}{\partial x^i} \qquad \text{on} \quad \Sigma \tag{2.16}$$

It is fairly obvious that a function ϕ^* satisfying (2.16) always exists and this will be assumed hereafter. [A proof can be given as in Bonnor (1969).]

ϕ^* will not satisfy (2.9) throughout D; where it does not, J^i is given by (2.7). In this way we can construct a globally regular electromagnetic field representing charges moving with speed c within D, and empty space outside.

3. *Model of a Charged Particle Moving with Speed c*

As an example let us consider the potential,

$$\phi = \psi(u)\left(2\log\frac{r}{a} + 1\right), \qquad r \geqslant a \tag{3.1}$$

† This is a satisfactory description of, for example, a stream of electrons. However it does not describe the state of affairs inside a conductor, where in every small region there exist positive and negative charges moving with different velocities. In the latter situation, the nullity of J^i would not imply that the charges travel with speed c.

$$\phi = \psi(u)\, r^2/a^2 \qquad r \leqslant a \tag{3.2}$$

where $r = +(x^2 + y^2)^{1/2}$, and $\psi(u)$ is of class C^2. Then ϕ is of class C^1 near $r = a$ and of class C^2 in any closed interval of r not containing $r = a$, and although ϕ tends to infinity with r, the field F_{ik} tends to zero, so (3.1) and (3.2) are physically acceptable solutions. The current four-vector is

$$J^i = 0, \qquad r > a \tag{3.3}$$

$$J^i = -(\pi a^2)^{-1}\, \psi(0, 0, 1, 1), \qquad r < a \tag{3.4}$$

If $\psi(u) = \text{const.}$, the source is a constant beam of charges moving with speed c. The field, given by (2.5), is the electric field of a line charge, together with the magnetic field of a steady straight current.

The charges of the beam do not interact, because the force term $F_{ik} J^k$ vanishes. Hence the beam can persist without non-electromagnetic forces. This applies too to the other solutions of Sections 3 and 4.

Now suppose that $\psi(u)$ is an even function with the shape of a single pulse [e.g. $\psi = \exp(-\frac{1}{2}u^2)$] satisfying

$$\psi \leqslant O(u^{-(1+\epsilon)}) \qquad \text{as} \qquad |u| \to \infty, \qquad \epsilon > 0 \tag{3.5}$$

so that

$$\int_{-\infty}^{\infty} \psi(u)\, du$$

exists. Then the source is a pulse of charge moving with speed c along the z-axis. This could be taken as the model of a charged particle. Let O be an observer at $P(x_0, y_0, z_0)$; according to (3.1), (3.2) and (2.5), O finds that the greatest values of F_{ik} occur at the maximum in ψ: that is, at just the moment when the thickest part of the pulse is passing P in its journey along Oz. As the pulse becomes sharper and sharper, O experiences an electromagnetic field for a shorter and shorter time; but (if P is outside the pulse) O finds during this short time the electric field of an infinite line charge, and the magnetic field of an infinite straight current.

This behaviour seemed to me so remarkable that I decided to see whether the vector potential (3.1) could be constructed from the current (3.4) by means of retarded potentials alone. It seemed quite possible at the outset that one might need to add either a contribution from the advanced potential, or a non-singular wave function, or both. In fact, as is shown in the Appendix, (3.1) *can* be constructed by retarded potentials alone, except for a function of $t - z$, which is unimportant because it vanishes when one forms the field by means of (2.4).

One can take, instead of (2.3)

$$\phi = \phi(x, y, v), \qquad v \stackrel{\text{def}}{=} t + z \tag{3.6}$$

In this case one finds that the field arises from a *purely advanced* potential. Since (3.6) arises from (2.3) simply by reversing the direction of the z-axis this seems rather strange. However, (3.6) also comes from (2.3) by altering the sign of t, so from this point of view a change from retarded to advanced is natural. However, it does suggest that for fields like these the distinction between advanced and retarded potentials is of no great importance.

The instantaneous value of the relative energy W of the exterior field (3.1) is obtained from (2.12):

$$W_{\text{ext}} = 4 \int_{-\infty}^{\infty} \int_{0}^{2\pi} \int_{a}^{\infty} r^{-1}[\Psi(t-z)]^2 \, dr \, d\theta \, dz \tag{3.7}$$

which diverges. We conclude that a pulse, however short, of charge of one sign moving with speed c is unphysical according to Maxwell's theory.

It will be shown in the next section that this conclusion need not apply if charges of both signs are present.

4. *A Dipole Particle Moving with Speed c*

Consider the potential

$$\phi = \psi(u) r^{-1} \cos\theta, \qquad r \geqslant a \tag{4.1}$$

$$\phi = \psi(u) a^{-1} \cos\theta \left[-3\left(\frac{r}{a}\right)^3 + 4\left(\frac{r}{a}\right)^2\right], \qquad r \leqslant a \tag{4.2}$$

where $x = r\cos\theta$, $y = r\sin\theta$ and $\psi(u)$ is a pulse function as described in Section 3. ϕ is of class C^1 at $r = a$ and of class C^2 elsewhere. For $r > a$, $J^i = 0$ and

$$J^i = \frac{3\psi\cos\theta}{\pi a^3}\left(\frac{2r}{a} - 1\right)(0, 0, 1, 1), \qquad r < a \tag{4.3}$$

We can take this as a model of a particle containing charge of both signs. The total charge, i.e.

$$\int_{-\infty}^{\infty} \int_{0}^{2\pi} \int_{0}^{a} \rho r \, dr \, d\theta \, dz$$

vanishes, but the particle has a dipole moment. The total electromagnetic energy is, from (2.12),

$$W_{\text{tot}} = \int_{-\infty}^{\infty} \int_{0}^{2\pi} \int_{0}^{\infty} \left(\frac{\partial\phi}{\partial r}\right)^2 r \, dr \, d\theta \, dz$$

which is *finite*. Since all quantities associated with the particle are well behaved, *it seems that Maxwell's theory allows a dipole particle travelling with speed c.*

5. *Conclusion*

The main conclusions are as follows.

(1) Globally regular solutions of Maxwell's equations exist for charge moving with the speed of light. However, if the net charge is not zero, the total energy is infinite. If the net charge is zero the total energy can be finite, and such solutions seem physically acceptable.

(2) This moving charge generates plane-fronted waves.

(3) Roughly speaking, a particle with a net charge moving with speed c generates at an exterior point P the field of a static, infinite line-charge, and of a steady straight current but only whilst it is passing P at its shortest distance.

Appendix: *Construction of Solution* (3.1) *by Retarded Potentials*

It is clear from (2.3) and (2.9) that ϕ satisfies d'Alembert's equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}+\frac{\partial^2\phi}{\partial t^2}=-4\pi\rho \tag{A.1}$$

and it is known (Ferraro, 1962) that every solution of this equation may be written

$$\phi(x,y,z,t)=\int\limits_V\frac{[\rho]}{R}\,dv+\frac{1}{4\pi}\int\limits_S\left\{\frac{1}{R}\left[\frac{\partial\phi}{\partial n}\right]-\frac{\partial}{\partial n}\left(\frac{1}{R}\right)[\phi]+\frac{1}{R}\frac{\partial R}{\partial n}\left[\frac{\partial\phi}{\partial t}\right]\right\}\,dS \tag{A.2}$$

where S is a surface enclosing volume V, $\partial/\partial n$ means differentiation along the outward normal to S, R is the distance from the source-point to the field-point (x,y,z,t) and $[K]$ denotes the value of K at time $t-R$. The first integral represents the retarded contribution from that part of the source which lies inside S, and the second represents fields having no sources inside S. The contribution from advanced potentials, if ϕ contains any, will be included in the surface integral; but the latter may also include other contributions, namely from sources outside S, and from source-free fields, such as plane waves.

The problem here is to see, if S is chosen an infinite sphere centre the origin, whether ϕ given by (3.1) arises only from the volume integral, or whether a surface integral is also necessary. To simplify the work I shall not use the interior solution (3.2), but instead I shall treat the source as a line with

$$\rho=-\psi(u)\,\delta(x)\,\delta(y) \tag{A.3}$$

which corresponds with (3.4). Let

$$I(x,y,z,t)=\int\limits_V\frac{[\rho]}{R}\,dv=-\int\limits_{-\infty}^{\infty}\frac{\psi(t-R-\zeta)}{R}\,d\zeta \tag{A.4}$$

where

$$R = +[x^2 + y^2 + (z-\zeta)^2]^{1/2} \tag{A.5}$$

We wish to find the difference between I, and ϕ given by (3.1). Integrating (A.4) by parts, we have

$$I = -\psi(t - R - \zeta)\log(R - z + \zeta)\Big|_{-\infty}^{\infty}$$

$$-\int_{-\infty}^{\infty} R^{-1}(R - z + \zeta)\log(R - z + \zeta)\,\psi'(t - R - \zeta)\,d\zeta \tag{A.6}$$

and inserting the limits we have

$$I = 2\psi(t-z)\log r + C\psi(t-z) - J \tag{A.7}$$

where J is the integral in (A.6) and C is an infinite constant;† we have assumed that $\psi(u)$ satisfies the very mild condition

$$\mathscr{L}t_{u\to\infty}\,\psi(-u)\log u = 0$$

which would be satisfied as a consequence of (3.5).

We next examine J. If we differentiate it, integrating by parts where necessary, we find, using (3.5)

$$\frac{\partial J}{\partial r} = 0, \qquad -\frac{\partial J}{\partial z} = \frac{\partial J}{\partial t}$$

so it follows that J is a function of $t - z$ only. Comparing (A.7) and (3.1) we find that both I and ϕ have form

$$2\psi(u)\log r + \text{function of } (t-z)$$

When we form the F_{ik} by (2.4) the functions of $t - z$ have no effect so we conclude that the exterior field of Section 3 arises from the retarded potential of the charge distribution (3.4).

References

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† This infinite integration constant arises in the same way as the one obtained during the calculation of the potential of an infinite static line charge. It is no cause for alarm because it disappears when one constructs the field by (2.4).